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Spatiotemporal correlation of colored noise

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We calculated the noise correlation in a Gaussian stochastic process that is non-δ-function-correlated in both space and time. The colored noise obeys a linear reaction-diffusion Langevin equation with Gaussian white noise. Our result is a generalization of the Ornstein-Uhlenbeck process to take into account finite correlation in space as well as in time for colored noise.

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I. INTRODUCTION

Many problems in nonequilibrium statistical mechanics are modeled by Langevin equations [1–6]. For instance, the application of Langevin equations to describe a nonequilibrium surface [7–13] and self-organized criticality [14,15] are recent active areas of research. In these equations, a stochastic term $\eta(\mathbf{r},t)$ is added to the macroscopic and deterministic equation, of the form

$$\frac{\partial\psi}{\partial t}=f([\psi(\mathbf{r},t)],\nabla\psi,\nabla^2\psi)+\eta(\mathbf{r},t) \ . \qquad (1)$$

Here $\psi(\mathbf{r},t)$ is the relevant variable of the system, and the first term on the right-hand side is a deterministic force. $\eta(\mathbf{r},t)$ is a random term called noise, which is usually assumed to be Gaussian and accounts for either internal degrees of freedom or fluctuations in the constraints imposed externally on the system. In the first case, the noise is called internal noise and is assumed to be white noise. That means the values of the random field in a given point and at a given time does not depend on its value in other points or at other times,

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r},t')\rangle=2\epsilon\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \ , \qquad (2)$$

where ϵ is the intensity of the noise and $\langle \ \rangle$ denotes an average over the probability distribution of the random field. In the case of external noise, or noise coming from fluctuations in the constraints imposed externally on the system, the correlations of the random field between different points and times could be nonzero. In this case, the spectrum of the noise in both momentum k and frequency ω is no longer constant, so one speaks of colored noise.

Colored noise has been studied extensively in first- and second-order processes. In first-order processes, it is known that colored noise is non-Markovian when the distribution between switches is not exponential. When the distribution between switches is exponential (in which case the colored noise is Markovian and is sometimes referred to as the random telegraph signal), the probability density function obeys an integro-differential equation that reduces to the telegrapher's equation in the driftless case [1,6,16,17]. Second-order processes driven by colored noise are of practical interest in many branches of physics and in engineering [18–22]. Recently, colored noise has also been used in nonequilibrium surface-growth models [9,13].

The most famous example of colored noise is the Ornstein-Uhlenbeck process, which is Gaussian and has zero mean and a correlation given by [6,23]

$$\langle \xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=\frac{\epsilon}{\tau}e^{-|t-t'|/\tau}\delta(\mathbf{r}-\mathbf{r}') \ . \qquad (3)$$

Here τ is the correlation time of the noise, i.e., a measure of its memory in time. The stochastic differential equation that governs its evolution is

$$\dot{\xi}(t)=-\frac{1}{\tau}\xi(t)+\frac{1}{\tau}\eta(t) \ , \qquad (4)$$

where $\eta(t)$ is a white noise following Eq. (2) without spatial dependence.

Recently, Garcia-Ojalvo, Sancho, and Ramirez-Piscina [24] proposed a generalization of this very simple idea to take into account finite correlation in space as well. The simplest stochastic differential equation modeling such a noise is the following reaction-diffusion equation:

$$\dot{\xi}(\mathbf{r},t)=-\frac{1}{\tau}(1-\lambda^2\nabla^2)\xi+\frac{1}{\tau}\eta(\mathbf{r},t) \ , \qquad (5)$$

where $\eta(\mathbf{r},t)$ is again a white noise following (2), and λ is the correlation length of $\xi(\mathbf{r},t)$. They had also proposed [24] a method of numerically generating colored noise according to (5).

The purpose of this paper is to calculate the correlation of the colored noise obeying (5) so as to obtain a generalization of the Ornstein-Uhlenbeck process (3) to the case with spatial correlations. In Sec. II we will derive expressions for these correlations for spatial dimensions $d=2$ and 3. Section III is a summary and conclusion.

II. SPATIALLY DEPENDENT COLORED NOISE

The spatial Fourier transform of (5) has the form

$$\frac{d\tilde{\xi}}{dt}=-\frac{1}{\tau}(1+\lambda^2k^2)\tilde{\xi}+\frac{1}{\tau}\tilde{\eta}(\mathbf{k},t)\ , \tag{6}$$

where the quantities with tildes denotes their spatial Fourier transforms,

$$\tilde{\xi}(\mathbf{k},t)=\int \xi(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}d^dr\ , \tag{7}$$

$$\tilde{\eta}(\mathbf{k},t)=\int \eta(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}d^dr\ . \tag{8}$$

Equation (6) has the solution

$$\tilde{\xi}(\mathbf{k},t)=\frac{1}{\tau}e^{-(1+\lambda^2k^2)t/\tau}\int_{-\infty}^{t}e^{(1+\lambda^2k^2)t'/\tau}\tilde{\eta}(\mathbf{k},t')dt'\ . \tag{9}$$

Using (9), we can calculate the correlation

$$\langle\tilde{\xi}(\mathbf{k},t)\tilde{\xi}(\mathbf{k}',t')\rangle=\frac{1}{\tau^2}e^{-(1+\lambda^2k^2)t/\tau}e^{-(1+\lambda^2k^2)t'/\tau}\int_{-\infty}^{t}du\int_{-\infty}^{t'}dv\ e^{(1+\lambda^2k^2)u/\tau}e^{(1+\lambda^2k'^2)v/\tau}\langle\tilde{\eta}(\mathbf{k},u)\tilde{\eta}(\mathbf{k}',v)\rangle\ . \tag{10}$$

From (2), the correlation for the white noise $\tilde{\eta}$ has the form

$$\langle\tilde{\eta}(\mathbf{k},t)\tilde{\eta}(\mathbf{k}',t')\rangle=2\epsilon(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')\delta(t-t')\ . \tag{11}$$

Using (11) in (10) we have

$$\begin{aligned}\langle\tilde{\xi}(\mathbf{k},t)\tilde{\xi}(\mathbf{k}',t')\rangle&=\frac{2\epsilon}{\tau^2}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')e^{-(1+\lambda^2k^2)(t+t')/\tau}\int_{-\infty}^{t}du\int_{-\infty}^{t'}dv\ e^{(1+\lambda^2k^2)(u+v)/\tau}\delta(u-v)\\&=\frac{2\epsilon}{\tau^2}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')e^{-(1+\lambda^2k^2)(t+t')/\tau}\int_{-\infty}^{\infty}\Theta(t-u)du\int_{-\infty}^{\infty}dv\ \Theta(t'-v)e^{(1+\lambda^2k^2)(u+v)/\tau}\delta(u-v)\\&=\frac{2\epsilon}{\tau^2}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')e^{-(1+\lambda^2k^2)(t+t')/\tau}\int_{-\infty}^{\infty}\Theta(t-u)\Theta(t'-u)e^{2(1+\lambda^2k^2)u/\tau}du\\&=\frac{\epsilon}{\tau}(2\pi)^d\delta(\mathbf{k}-\mathbf{k}')e^{-|t-t'|/\tau}\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2}\ .\end{aligned} \tag{12}$$

The correlation $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle$ is given by the inverse Fourier transform of (12),

$$\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=\frac{1}{(2\pi)^d}\frac{\epsilon}{\tau}e^{-|t-t'|/\tau}\int d^dk\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\ . \tag{13}$$

We will calculate the integral in (13) for spatial dimensions $d=2$ and 3 separately. For $d=2$, we have

$$\begin{aligned}\int d^2k\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}&=\int_0^{2\pi}d\theta\int_0^{\infty}\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k_2}e^{-ik|\mathbf{r}-\mathbf{r}'|\cos\theta}k\ dk\\&=\int_0^{\infty}\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2}J_0(k|\mathbf{r}-\mathbf{r}'|)k\ dk\\&=\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda^2)^n(-1)^ne^{-\lambda^2k^2|t-t'|/\tau}J_0(k|\mathbf{r}-\mathbf{r}'|)k^{2n+1}dk\ .\end{aligned} \tag{14}$$

Using Eq. (6.631.4) of Gradshteyn and Ryzhik [25], the integral in (14) can be calculated,

$$\begin{aligned}\int d^2k\frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}&=\sum_{n=0}^{\infty}\lambda^{2n}(-1)^n\frac{(\tau|\mathbf{r}-\mathbf{r}'|)^{2n}\tau}{2(\lambda^2|t-t'|)^{2n+1}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2\tau}{4\lambda^2|t-t'|}\right]\\&=\frac{2\tau|t-t'|}{4\lambda^2|t-t'|+\tau^2|\mathbf{r}-\mathbf{r}'|^2}\exp\left[\frac{|\mathbf{r}-\mathbf{r}'|^2\tau}{4\lambda^2|t-t'|}\right]\ .\end{aligned} \tag{15}$$

Therefore for $d=2$, using (13) and (15), we obtain for the correlation

$$\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t)\rangle=\frac{1}{2\pi^2}\frac{\epsilon|t-t'|}{4\lambda^2|t-t'|^2+\tau^2|\mathbf{r}-\mathbf{r}'|^2}\exp\left[-\frac{4\lambda^2|t-t'|^2+\tau^2|\mathbf{r}-\mathbf{r}'|^2}{4\lambda^2|t-t'|\tau}\right]\ . \tag{16}$$

For spatial dimension $d=3$, we have, for the integral in Eq. (13), using Eq. (3.954.1) of [25],

$$\int d^3k \frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2} e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = 4\pi \int_0^\infty \frac{e^{-\lambda^2k^2|t-t'|/\tau}}{1+\lambda^2k^2} k \sin(k|\mathbf{r}-\mathbf{r}'|)dk$$
$$= -\frac{\pi^2}{\lambda^2} e^{|t-t'|/\tau} \left\{ 2\sinh(|\mathbf{r}-\mathbf{r}'|/\lambda) + e^{-|\mathbf{r}-\mathbf{r}'|/\lambda} \Phi\left[\frac{2\lambda|t-t'|-\tau|\mathbf{r}-\mathbf{r}'|}{2\lambda\sqrt{\tau|t-t'|}}\right] - e^{|\mathbf{r}-\mathbf{r}'|/\lambda}\Phi\left[\frac{2\lambda|t-t'|+\tau|\mathbf{r}-\mathbf{r}'|}{2\lambda\sqrt{\tau|t-t'|}}\right]\right\}, \tag{17}$$

where $\Phi(x) \equiv 2/\sqrt{\pi}\int_0^x e^{-t^2}dt$ is the probability integral. Therefore for $d=3$, using (13) and (17), the noise correlation has the form

$$\langle \xi(\mathbf{r},t)\xi(\mathbf{r},t')\rangle = -\frac{\epsilon}{8\tau\lambda^2}\left\{2\sinh(|\mathbf{r}-\mathbf{r}'|/\lambda) + e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}\Phi\left[\frac{2\lambda|t-t'|-\tau|\mathbf{r}-\mathbf{r}'|}{2\lambda\sqrt{\tau|t-t'|}}\right] - e^{|\mathbf{r}-\mathbf{r}'|/\lambda}\Phi\left[\frac{2\lambda|t-t'|+\tau|\mathbf{r}+\mathbf{r}'|}{2\lambda\sqrt{\tau|t-t'|}}\right]\right\}. \tag{18}$$

Using the asymptotic expansion of $\Phi(x)$ for large x,

$$\Phi(x) \simeq 1 - \frac{1}{\pi}\frac{1}{x}e^{-x^2}, \tag{19}$$

the noise correlation for $d=3$ reduces to

$$\langle \xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle \simeq \frac{\epsilon\sqrt{\tau}}{4\lambda\pi^{3/2}} \frac{|\mathbf{r}-\mathbf{r}'|\sqrt{|t-t'|}}{4\lambda^2|t-t'|^2+\tau^2|\mathbf{r}-\mathbf{r}'|^2} \exp\left\{-\frac{4\lambda^2|t-t'|^2+|\mathbf{r}-\mathbf{r}'|^2\tau^2}{4\lambda^2\tau|t-t'|}\right\}, \tag{20}$$

for $|t-t'| \gg 1$.

III. SUMMARY AND CONCLUSION

We have calculated the correlations in colored noise obeying a linear reaction-diffusion Langevin equation [Eq. (5)]. This is a generalization of the Ornstein-Uhlenbeck process to take into account finite correlations in space as well as in time. Our result is given by (16) for spatial dimension $d=2$ and by (18) for $d=3$. For large values of $|t-t'|$, (18) reduces to (20), which is very similar in form to the two-dimensional case (16). For $\lambda \to 0$, the $|\mathbf{r}-\mathbf{r}'|$ dependences in (16) and (18) reduce to a δ function and we obtain the original Ornstein-Uhlenbeck correlation (3).

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